Anisotropic Plasticity Constitutive Law for Sea Ice

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LONG-TERM GOALS

My long term goal is to develop sea ice dynamics models that describe behavior on kilometer scales and larger, base these models on the smaller scale physical processes known to control leading, rafting and ridging, and implement these models in the next-generation Polar Ice Prediction System (PIPS).

OBJECTIVES

An anisotropic plasticity constitutive law is being developed to describe and forecast ice stress, deformation, lead direction, and ice condition at scales from a few kilometers to hundreds of kilometers. The key difference between this anisotropic plasticity model and isotropic models is that it can describe the formation and orientation of new lead or ridge systems and track their thickness distributions. The work is divided into several task areas: describe all elements of the constitutive law; implement the law in computer codes to test its performance; describe both the ice dynamics behavior and the noise generated by the moving ice; transition the model into Navy systems such as PIPS; and publish the results.

APPROACH

Anisotropic constitutive Law. *Coon, et al.* [1998] and *Pritchard* [1998a] introduced an anisotropic elastic-plastic constitutive law, with a yield surface composed of individual surfaces for each lead, and strengths dependent on ice conditions in all leads. *Pritchard* [1998a] introduced an oriented thickness distribution ice conditions as a function of thickness and direction, with isotropy defined as a state where ice conditions are identical in all orientations. The constitutive law requires a numerical integration [*Pritchard*, 1998b]. It is being implemented in a finite element code that uses the explicit leapfrog scheme. This scheme is useful for short-term simulations and forecasts. An implicit numerical scheme is being adapted for long-term integrations needed to conduct climate simulations.

Time Scales of Elastic-plastic and Viscous-plastic Models. The time scales that naturally occur in sea ice dynamics models are being studied [*Pritchard*, 2001]. Ice inertia is negligible when the daily average behavior is simulated. This term has been removed from the model to study quasi-steady behavior. The elastic-plastic law still has an e-folding time scale on the order of a day. This result suggests that the elastic-plastic and viscous-plastic laws should be used to simulate behavior on different time scales. The models are being analyzed to demonstrate this assertion.

Elastic-plastic and viscous-plastic constitutive laws are being compared to understand their behavior, and to learn more about their similarities and differences. In the past, we focused on the fact that

different laws were used to achieve closure when stress is within the yield surface. I now believe this is a red herring, and have begun to compare behavior during plastic flow when stress is on the yield surface. To understand this point, consider the stress history at a point when a constant stretching tensor $\mathbf{D}$ is applied. The elastic-plastic law responds over time. Stress $\boldsymbol{\sigma}$ begins at its initial state, moves to the yield surface, and then slides along the yield surface until a steady state is reached. At the final stress state, the (co-rotational) stress and elastic strain rates are zero and the plastic stretching equals the total stretching $\mathbf{D}_p = \mathbf{D}$. The stress and plastic stretching must also satisfy the yield constraint $\phi(\boldsymbol{\sigma}, p^*) = 0$ and the flow rule $\mathbf{D}_p = \partial\phi/\partial\boldsymbol{\sigma}$. This is the same stress state that is instantaneously reached by the viscous-plastic law. Thus, viscous-plastic behavior is the quasi-steady limit of elastic-plastic behavior.

All previous models have mixed components with different time scales, but here I describe two different consistent models. For synoptic scale simulations, ice inertia, tidal oscillations, and inertia of the upper ocean layer are important. A fully dynamic momentum equation and an active upper ocean layer are needed [*Hibler, et al.*, 1998]. An elastic-plastic law is needed to follow the stress path. For daily or longer resolution, ice inertia, tidal oscillations, and inertia of the upper ocean layer are not important. A quasi-steady momentum balance and passive water drag law should be used. A viscous-plastic law describes behavior when the stress path is not important.

Stability Analysis. At the International Union of Theoretical and Applied Mechanics Conference held at the University of Alaska, 13-16 June 2000, ice dynamics modelers were challenged to ensure that all simulations are well posed, and that models are stable and convergent. I have taken this challenge seriously, and have formulated an approach that can be used to analyze all ice dynamics models [*Pritchard*, in preparation]. It extends the methods used by *Gray* [1999] and *Schreyer* [2000].

For the elastic-plastic law, stability is assured by satisfying Drucker's Postulate, and is not discussed here. For the viscous-plastic law, no such general approach exists, and a new one is introduced. I limit this presentation to a quasi-steady viscous-plastic model with ocean currents neglected. The ice momentum balance is

$$\mathbf{b}(\mathbf{v}) = \boldsymbol{\tau} + \nabla \cdot \boldsymbol{\sigma} \tag{1}$$

where $\mathbf{b}$ includes terms that depend on velocity (Coriolis force and water drag), and $\boldsymbol{\tau}$ includes terms independent of the velocity solution (air stress and sea surface tilt). The internal stress is $\boldsymbol{\sigma}$. If the isotropic viscous-plastic law uses an elliptical yield surface, and normal flow rule, and a replacement closure for which zero stretching gives zero stress, then it has the form [e.g., *Hibler*, 1979; *Ip*, 1993]

$$\frac{1}{\zeta}\boldsymbol{\sigma} = \frac{2}{e^2}\mathbf{D} + \left(1 - \frac{1}{e^2}\right)D_I\mathbf{1} - \Delta\mathbf{1} \tag{2}$$

where e is the ratio of pressure to shear stress axes of the ellipse, $\Delta^2 = D_I^2 + (D_{II}/e)^2$ is a stretching rate, and the bulk modulus can have the form $\zeta = p^*/2\max(\Delta, \Delta_{min})$, and Δ_{min} is a constant. A simple hardening law is assumed during plastic flow, where the rate of change of strength $\dot{p}^*$ depends only on the rate of area change D_I

$$\dot{p}^* = -C_h D_I. \tag{3}$$

where C_h is the hardening rate.

A linear perturbation is introduced to analyze stability of the system of equations. The quasi-steady momentum equation perturbation is

$$\mathbf{b}' \cdot \delta \mathbf{v} = \nabla \cdot \delta \boldsymbol{\sigma} \tag{4}$$

where $\delta \mathbf{v}$ is the velocity perturbation and $\delta \boldsymbol{\sigma}$ is the stress perturbation. We interchange the time and space derivatives with the perturbation operation. The body force tensor satisfies

$$\mathbf{b}' = mf\mathbf{B}_{\pi/2} + \rho_w C_w \mathbf{B}_w \cdot \left(\frac{\mathbf{v} \otimes \mathbf{v}}{|\mathbf{v}|} + |\mathbf{v}|\mathbf{1} \right). \tag{5}$$

Forcing by wind and thermal perturbations are ignored because we seek nontrivial solutions to the homogeneous system. When $\zeta = p*/2\Delta$, the perturbation to the viscous-plastic law is

$$2\Delta\delta\boldsymbol{\sigma} + 2\delta\Delta\boldsymbol{\sigma} = p* \left[\frac{2}{e^2}\delta\mathbf{D} + \left(1 - \frac{1}{e^2}\right)\delta D_I \mathbf{1} - \delta\Delta\mathbf{1} \right] + \delta p* \left[\frac{2\Delta\boldsymbol{\sigma}}{p*} \right] \tag{6}$$

and the perturbation to the hardening law is

$$\delta\dot{p}* = -C_h \delta D_I . \tag{7}$$

For solutions to the homogeneous problem, we seek to determine the characteristic equation by introducing modal solutions for the velocity, stress, and strength perturbations. For example, the velocity perturbation has the form

$$\delta\mathbf{v} = \delta\overline{\mathbf{v}} e^{\frac{ik}{L}\mathbf{n}\cdot\mathbf{x} - \kappa t} \tag{8}$$

which varies as a sinusoid in the direction of unit vector $\mathbf{n}$ and decays at rate κ. The length L indicates domain size and k is the integer mode number. The coefficients are constant in the solution to the linearized perturbation equations. The modal solutions must be studied for all direction vectors $\mathbf{n}$. Complex decay coefficients represent propagating waves. The real part of κ cannot be negative if solutions are to be stable. The spatial gradient of the modal velocity is $\nabla\delta\mathbf{v} = \frac{ik}{L}\delta\mathbf{v}\otimes\mathbf{n}$, its time derivative is $\delta\dot{\mathbf{v}} = -\kappa\delta\mathbf{v}$, and the stress divergence is $\nabla\cdot\delta\boldsymbol{\sigma} = \frac{ik}{L}\delta\boldsymbol{\sigma}\cdot\mathbf{n}$. Each governing equation can be expanded into normal modes, and common exponential terms removed to give relationships between the modal coefficients $\delta\overline{\mathbf{v}}$, $\delta\overline{\boldsymbol{\sigma}}$, and $\delta\overline{p}*$. Expressing the algebraic equations in matrix form gives

$$\begin{pmatrix} b'_{xx} + \left(\frac{k}{L}\right)^2 A_{xx} & b'_{xy} + \left(\frac{k}{L}\right)^2 A_{xy} & \left(\frac{k}{L}\right)^2 t_x \\ b'_{yx} + \left(\frac{k}{L}\right)^2 A_{yx} & b'_{yy} + \left(\frac{k}{L}\right)^2 A_{yy} & \left(\frac{k}{L}\right)^2 t_y \\ C_h n_x & C_h n_y & -\kappa \end{pmatrix} \begin{pmatrix} \frac{ik}{L}\delta\overline{v}_x \\ \frac{ik}{L}\delta\overline{v}_y \\ \delta\overline{p}* \end{pmatrix} = 0 \tag{9}$$

where the second order symmetric viscous-plastic acoustic tensor is

$$\mathbf{A}_{vp} = \frac{p*}{2\Delta}\left[\frac{1}{e^2}\mathbf{1} + \mathbf{n}\otimes\mathbf{n} - (2\mathbf{t}+\mathbf{n})\otimes(2\mathbf{t}+\mathbf{n}) \right] \tag{10}$$

and $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}/p*$ is a traction vector.

The characteristic polynomial results from setting the determinant of the coefficients of the homogeneous system of equations to zero. This defines the condition when nontrivial modal solutions can exist under zero forcing. It is a linear polynomial in decay rate κ. If the real parts of these roots are positive, then the modes decay with time, and behavior is stable. If they are negative, then the modes grow with time, and the behavior is unstable. Thus, we have described a method for studying the stability of quasi-steady, isotropic, viscous-plastic models. Although final results are not yet available, it appears that some unstable modes exist even during closing [*Pritchard*, in preparation]. Unfortunately, this result is more serious than that of *Gray* [1999], who found unstable behavior during uniaxial opening. The method can be generalized to study the stability of all models, whether dynamic or quasi-steady, isotropic or anisotropic, and elastic-plastic or viscous-plastic.

Numerical Solution Scheme. The quasi-steady model is solved using an implicit numerical scheme [*Pritchard*, 2001b]. The original scheme solved for the velocity vector using Newton's method. During the past year, I have adapted the GMRES (Generalized Minimum Residual) method [*Kelley*, 1995] to solve the linear inner set of equations in Newton's method without the need for determining the Jacobian matrix [*Pritchard*, in preparation]. This method is useful for anisotropic models because their Jacobian matrices are not readily derived. The method has been tested with *1d* simulations in which both velocity components are nonzero using elastic-plastic and viscous-plastic models.

WORK COMPLETED

All components of the anisotropic model have been developed, including the elastic-plastic law (yield surface, flow rule, elastic closure and kinematic relationship) and the oriented thickness distribution. Realistic choices are available for describing when new leads form, and their orientations. A scheme for integrating the stress state over a time step is available, and several numerical schemes for conducting *2d* simulations have been studied. *Pritchard* [2001] showed that ice inertia can be removed from the system of equations, and the quasi-steady elastic-plastic model can be used for long-term simulations. That work also suggests that the implicit numerical schemes are not much more efficient than the explicit leapfrog scheme for elastic-plastic models.

An approach has been formulated to analyze the models; it extends the methods used by *Gray* [1999] and *Schreyer* [2000]. A linear perturbation about the solution is analyzed by introducing normal modes of the form $e^{\frac{ik}{L}\mathbf{n}\cdot\mathbf{x}-\kappa t}$. The real part of the decay rate κ must not be negative if solutions are bounded and convergent. This is required for well posed problems.

The implicit numerical scheme will be solved using a modern GMRES (Generalized Minimum Residual) method. Idealized *1d* simulations have been conducted to test the new implicit numerical method using both elastic-plastic and viscous-plastic laws. The GMRES method is essential for anisotropic laws for which the Jacobian is too difficult to determine.

RESULTS

A new perspective on plasticity models has emerged. It is that viscous-plastic is the quasi-steady limit of elastic-plastic behavior, which leads to a better understanding of the appropriateness of each formulation. This identifies two different consistent models that are appropriate for solving problems having two different time scales. The elastic-plastic models are better used for short-term simulations and forecasts where time is resolved to synoptic scales, tidal oscillations are included, upper ocean

inertia is included, and the stress path is significant for determining the orientation of new leads. The viscous-plastic models are better used for long-term simulations resolved to daily and longer scales, inertia of ice and upper ocean are neglected, tides are neglected, and ice conditions are less well-resolved.

Quasi-steady behavior of viscous-plastic models has been shown to be unstable in a wide variety of cases [*Pritchard*, in preparation]. This is unfortunate because more work is needed to learn how to modify the constitutive law to ensure stability.

IMPACT/APPLICATIONS

Treating sea ice as an anisotropic material allows us to describe explicitly whether or not leads exist in a region and their orientations. Anisotropic constitutive laws describe the known behavior of sea ice more realistically. Perhaps accuracy will be most improved in lower resolution models because the discontinuous behavior will be better described within each grid cell.

The new perspective of viscous-plastic behavior as the quasi-steady limit of elastic-plastic behavior shows that each model is useful for solving problems having different time scales. This should lead to two different models being used for synoptic scale and long-term simulations.

The implicit numerical scheme allows long-term simulations to be conducted using both elastic-plastic and viscous-plastic constitutive laws. Understanding material stability is essential if we are to develop convergent numerical schemes.

TRANSITIONS

The anisotropic constitutive law will be useful for describing the formation and evolution of leads. This will provide information for submariners to surface safely, and for surface ship operators to optimize route selection.

The quasi-steady model formulation gives an alternative formulation for long-term climate studies and possibly for PIPS.

The implicit numerical scheme with the GMRES solver is useful for all models: elastic-plastic or viscous-plastic, fully dynamic or quasi-steady, and isotropic or anisotropic. However, an explicit leapfrog scheme seems just as good for fully dynamic elastic-plastic models, both isotropic and anisotropic. The methodology developed to study model stability will be useful for others who introduce new constitutive laws. All models must be stable if they are to describe well posed problems and are to provide convergent numerical solutions.

RELATED PROJECTS

I have been informally collaborating with several colleagues: J. M. N. T. Gray, U. Manchester, to study stability of viscous-plastic models; I. V. Polyakov, U. Alaska Fairbanks, to use an explicit coupled elastic-plastic ice-ocean model; H. L. Schreyer, to learn how to ensure that models are stable; L.- B. Tremblay, LDEO, Columbia U., to conduct idealized modeling studies comparing results of the anisotropic plasticity model with those of other models; M. D. Coon, to understand the anisotropic constitutive law better; and R. H. Bourke, NPS, to incorporate ambient noise models into PIPS.

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